Reply to "Comment on 'Intermittency in chaotic rotations'"

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We respond to the Comment by Pikovsky and Rosenblum by presenting physical intuitions, further arguments, and numerical results for the occurrence of intermittency in chaotic rotations.

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Intermittency and chaotic rotations are interesting recent topics in nonlinear dynamics, and we are pleased that our contribution [2] is of sufficient interest to have generated the preceding Comment by Pikovsky and Rosenblum [1]. We welcome the opportunity to respond and to clarify our work.

The main point of our original paper is that a chaotic rotation typically exhibits an intermittent behavior [2]. The physical intuition for our study comes from the observation of rotational structures in well studied chaotic flows such as the Rössler oscillator [3]. The differential equations of the Rössler system apparently contain an ideal rotational structure generated by a harmonic oscillator, as follows:

$$\frac{d\hat{x}}{dt} = -\omega_0 \hat{y},$$

$$\frac{d\hat{y}}{dt} = \omega_0 \hat{x}, \tag{1}$$

where $\hat{x}$ and $\hat{y}$ are the position and the velocity of the oscillator, respectively, and ω_0 is the frequency of the rotation. For the Rössler oscillator, if one examines the time evolution of the phase variable defined by $\phi(t) = \tan^{-1}[y(t)/x(t)]$, where x and y are dynamical variables of the Rössler equations, one finds that, on average, it increases approximately linearly with time: $\langle \phi(t) \rangle \sim \omega_0 t$ [Fig. 1(c) in [2]]. When the instantaneous frequency $\omega(t) = d\phi(t)/dt$ is examined, one finds that it tends to spend intervals of times near ω_0, with occasional bursts away from it [Fig. 1(d) in [2]]. The ideal rotation of the harmonic oscillator thus serves, in an approximate sense, as an "invariant" structure embedded in the corresponding chaotic rotation. If one regards rotations near the "invariant" one as the "off" state and those away from the "invariant" one as the "on" state, one clearly encounters a situation similar to that of on-off intermittency.

For the detection of on-off intermittency, there exists no rigorous criterion. Most existing works cite the following conditions [4,5]: (1) there exists an invariant state ("off" state) in which the trajectory can stay for various time intervals, (2) these time intervals are random, and (3) the trajectory can occasionally burst out of the "off" state. Figures 1(d), 2(d), and 3 in our paper [2] contain these three features. The authors of the Comment stated that a simple visual inspection of the time course of the instantaneous frequency does not lead to a recognition of on-off intermittency. Here we wish to point out the importance of looking at large time scales when performing such a visual inspection. The time scales should be such that some key quantities associated with the rotational dynamics can be treated effectively as random variables. Specifically, in Eq. (1) in our original paper [2], the quantities $\alpha(t)$ and $\beta(t)$ are deterministic, but they are effectively random variables on large time scales because they depend on dynamical variables of a chaotic system. To obtain a better idea about the behavior of $\alpha(t)$ and $\beta(t)$, we show in Figs. 1(a)–1(d) their probability distributions, where Figs. 1(a) and 1(c) are on a linear scale and Figs. 1(b) and 1(d) are on a semilogarithmic scale. Apparently, both $\alpha(t)$ and $\beta(t)$ have *approximately* zero mean and they are distributed in ranges near zero. Ideally, if $\alpha(t)$ and $\beta(t)$ are independent of $\omega(t)$, then Eq. (1) in our original paper [2] represents the standard setting for observing on-off intermittency [4,5]. Realistically, $\alpha(t)$ and $\beta(t)$ are related to $\omega(t)$, as the authors of the Comment pointed out. However, on large time scales, statistical independence is expected, at least in an approximate sense, as we argued in our original paper [2] [Eqs. (3) and (4) and the accompanying reasoning]. The main point here is that, on short time scales, no dynamical variable, if it comes from a deterministic, continuous-time flow, would look intermittent. However, on long time scales, some observables of a chaotic system, such as the instantaneous rotational frequency, can exhibit an intermittentlike behavior in the approximate sense described above. That is all we wished to convey in our original paper [2].

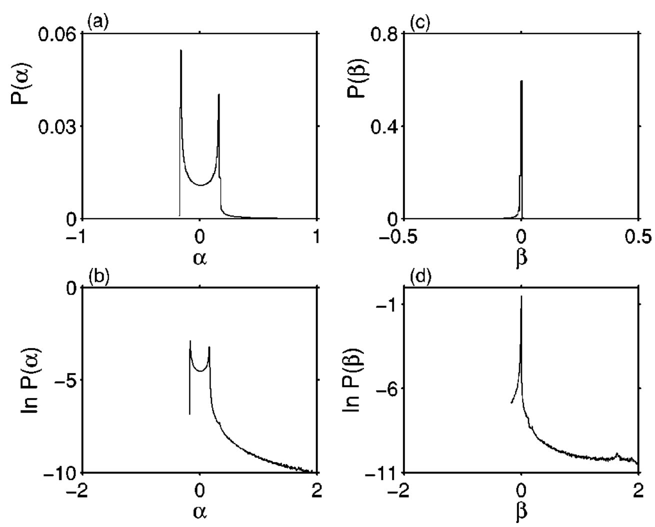

FIG. 1. Statistical distributions of $\alpha(t)$ and $\beta(t)$. (a,c) on a linear scale, (b,d) on a semilogarithmic scale.

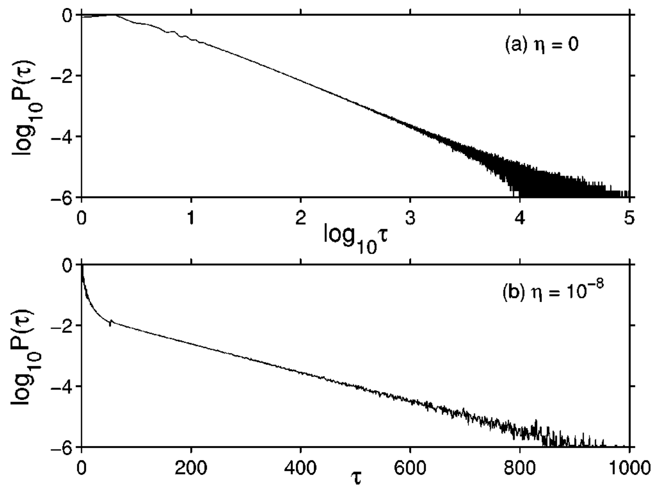

FIG. 2. For the simple model Eq. (2) for $a=2.75$, (a) for $\eta=0$, power-law (algebraic) distribution of the return times in y_n, the on-off intermittent variable, and (b) for $\eta=10^{-8}$, exponential distribution of return times.

The general theoretical argument against intermittency in chaotic rotations presented by the authors of the Comment appears to be the following: a process cannot be called ''intermittent'' if the return times between bursts are exponentially distributed. We wish to state that this argument is misleading, as the distribution of return times in a realistic situation of on-off intermittency is typically exponential at large times. It was previously pointed out by Heagy *et al.* [5] that the power-law distribution of return times is expected only at the onset of on-off intermittency for systems with a perfect invariant subspace (typically due to symmetry) that contains the ''off'' state. An exponential tail immediately develops as soon as the parameter moves away from the onset, although this tail may occur at such large times that it may not be observable in practice. A relevant but more important question is the following: In a realistic physical situation where there is a symmetry breaking so that there is no perfect invariant subspace, can one expect a power-law distribution? Our answer is generally no and, to the contrary, when a symmetry-breaking perturbation is present, no matter how small, the distribution of the return times immediately becomes exponential. This result can be illustrated by utilizing the following paradigmatic map model [5] for on-off intermittency, incorporating symmetry-breaking perturbations:

$$x_{n+1}=f(x_n),$$

$$y_{n+1}=ax_ny_n(1-y_n)+\eta, \tag{2}$$

where $f(x)$ is a chaotic map, η is the symmetry-breaking parameter, and for $\eta=0$ there is an invariant subspace defined by $y=0$. For $\eta \gtrsim 0$, on-off intermittency is observed. For $\eta=0$, the distribution of the return time τ (or laminar phase) appears to obey a power law, as shown on a logarithmic scale in Fig. 2(a), where $a=2.75$ and $f(x)$ is chosen to be the tent map (in this setting, the onset of on-off intermittency occurs at $a_c=e\approx 2.718\,28$). However, when η is increased from zero, the time series $\{y_n\}$ still looks quite on-off intermittent, but the original power-law distribution is immediately replaced by exponential ones at large times, as shown in Fig. 2(b) for the case where $\eta=10^{-8}$. A thorough analysis of such perturbed on-off intermittency has been performed recently [6]. The message here is that on-off intermittency with exponential distribution of return times is typical, since there exists no perfect symmetry and/or invariant subspace in physical situations. In the case of chaotic rotations, the term $\beta(t)$ in Eq. (1) in our paper [2] is a symmetry-breaking term, as we explained there. Thus, generally we expect to have an exponential distribution for large return times, although this exponential tail may or may not be observed in specific numerical examples.

We wish to remark that in dynamical systems with an invariant subspace the process of on-off intermittency can be viewed conveniently as a codimension-1 bifurcation [4] with parameter, say, p. That is, the intermittency usually exhibits a parameter dependence where the average length τ of the laminar phases diverges at a critical parameter p_c. In the presence of symmetry breaking (characterized by a parameter ϵ), the problem of the intermittency transition becomes a phenomenon of codimension-2 bifurcation. In such a general case, divergence of τ can still be expected in the two-dimensional parameter space (p,ϵ) for $p\to p_c$ and $\epsilon\to 0$ [6]. For a chaotic rotation, if these two parameters can be identified, then it may be possible to observe the divergence of τ. How to do this is an open problem.

Another feature of the effect of symmetry-breaking perturbations is the broadening of the ''off'' state, which is particularly apparent in the Lorenz attractor [7], as shown in Fig. 2 of our original paper [2]. Chaotic rotations in the Lorenz attractor are more complicated than those in the Rössler system, as the Lorenz attractor consists of two scrolls with distinct centers of rotation. Depending on the specific choice of the change of variables to obtain well-defined rotations [8], the degree of the intermittent behavior from visual inspection may vary. In any case, when a proper, broadened ''off'' state is defined and attention is focused on the behavior of the instantaneous frequency away from the '''off'' state, we observe, approximately, a behavior that we choose to call intermittency (to be consistent with the term used for the Rössler oscillator). The authors of the Comment did not agree with this usage of the terminology.

As for the relation between phase diffusion and intermittency, we can only remark that, while normal versus anomalous diffusions can in fact be rigorously defined in terms of the scaling laws and exponents of various statistical quantities, it is not possible to do so for intermittency. In particular, it is not proper to define intermittency based on scaling laws, such as the distribution of the return times, because such a distribution for on-off intermittent processes can be either exponential, or algebraic, or a mixture of both on different time scales. Thus, to argue for the absence of intermittency in terms of phase diffusion is meaningless. As far as Ref. [9] is concerned, we point out that the numerically observed fractional Brownian motion there is *not* associated with the overall phase of the Lorenz attractor. It is rather for certain specific rotational components obtained via an empirical mode decomposition procedure [8]. The authors of the Comment have misinterpreted the meaning of the phase variables utilized in Ref. [9].

In summary, so far there exists no rigorous definition of ‘‘intermittency’’ and, as the authors of the Comment have pointed out, determining whether there is intermittency depends mainly on visual examination. In the case of chaotic rotations, however, we believe that intermittency, at least in an approximate sense, can be claimed based on visual inspection on large time scales, physical intuition, and analysis of simple models.

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[1] A. Pikovsky and M. Rosenblum, preceding paper, Phys. Rev. E **64**, 058203 (2001).

[2] Y.-C. Lai, D. Armbruster, and E. J. Kostelich, Phys. Rev. E **62**, R29 (2000).

[3] O. E. Rössler, Phys. Lett. **71A**, 155 (1979).

[4] See, for example, E. A. Spiegel, Ann. N.Y. Acad. Sci. **617**, 305 (1981); H. Fujisaka and T. Yamada, Prog. Theor. Phys. **74**, 919 (1985); **75**, 1087 (1986); H. Fujisaka, H. Ishii, M. Inoue, and T. Yamada, *ibid.* **76**, 1198 (1986); L. Yu, E. Ott, and Q. Chen, Physica D **53**, 102 (1992); N. Platt, E. A. Spiegel, and C. Tresser, Phys. Rev. Lett. **70**, 279 (1993); F. Rödelsperger, A. Čenys, and H. Benner, *ibid.* **75**, 2594 (1995); Y.-C. Lai and C. Grebogi, Phys. Rev. E **52**, R3313 (1995).

[5] J. F. Heagy, N. Platt, and S. M. Hammel, Phys. Rev. E **49**, 1140 (1994).

[6] D. Marthaler, D. Armbruster, Y.-C. Lai, and E. J. Kostelich, Phys. Rev. E **64**, 016220 (2001).

[7] E. N. Lorenz, J. Atmos. Sci. **20**, 130 (1963).

[8] While special changes of variables can sometimes be found for defining proper phase variables for chaotic attractors with no unique centers of rotation in the phase space, such as the Lorenz attractor, a general methodology consists of measuring a chaotic signal, decomposing the signal into components with proper rotational structure, and performing the Hilbert transform to obtain an analytic signal. Specifically, note that for a proper rotation of a particle on a circle, $s(t)$, the projection of the particle trajectory on an arbitrary, diametral axis contains equal numbers of zeros and maxima and minima on time scales much larger than the period of the rotation. For a chaotic signal $x(t)$, the mode decomposition procedure then consists of the following three steps [10]: (1) construct two smooth splines connecting all the maxima and minima of $x(t)$ to yield $x_{max}(t)$ and $x_{min}(t)$, respectively; (2) compute $\Delta x(t) \equiv x(t) - [x_{max}(t) + x_{min}(t)]/2$; and (3) repeat steps 1 and 2 for $\Delta x(t)$ until the resulting signal corresponds to a proper rotation. Denote this signal by $C_1(t)$, which is the first component of $x(t)$. The difference $x_1(t) \equiv x(t) - C_1(t)$ is taken and steps 1–3 are repeated to yield the second component $C_2(t)$ from $x_1(t)$. The procedure continues until the component $C_M(t)$ shows no apparent time variation [10]. The original signal $x(t)$ can thus be expressed as $x(t) = \Sigma_{j=1}^{M} C_j(t)$, where the functions $C_j(t)$ are nearly orthogonal to each other [10]. By the nature of the decomposition procedure, the first component $C_1(t)$ corresponds to the fastest time variation of $x(t)$ and, hence, the signal has the smallest time scale. As the mode index j increases, the time scale increases and the mean frequency of the rotation decreases. Because each component $C_j(t) (j = 1, \ldots, M)$ corresponds to a proper rotation, the Hilbert transform can be applied to yield the following complex analytic signal: $\Psi_j(t) = C_j(t) + iH[C_j(t)]$, where $H[C_j(t)]$ is the Hilbert transform of $C_j(t)$. The phase variable $\phi_j(t) = \tan^{-1}[H[C_j(t)]/C_j(t)]$ can then be obtained. This methodology has been applied to the Lorenz system to address the phase characterization of chaotic attractors [9], and has also been employed to investigate transition to chaos in deterministic flows from the standpoint of rotation [11].

[9] T. Yalcinkaya and Y.-C. Lai, Phys. Rev. Lett. **79**, 3885 (1997).

[10] N. Huang, Z. Shen, S. R. Long, M. C. Wu, H. H. Shih, Q. Zheng, N.-C. Yeu, C. C. Tung, and H. H. Liu, Proc. R. Soc. London, Ser. A **454**, 903 (1998).

[11] Y.-C. Lai, Phys. Rev. E **58**, R6911 (1998).